Efimov I: X - separated scheme of finite type over a field k.

$D^b_{coh}(X) = D^b(\mathrm{coh}\, X)$ - DG category defined up to quasi equivalence

$\cup$

$\mathrm{Perf}(X) = \{ E \mid \text{locally } E \cong F \text{ - bounded complex of free sheaves of finite rank} \}$

Thm: $D(X) = D(A_X)$ X - unbounded derived category of quasi-coherent

$A = R\mathrm{Hom}(E, E)$ $E \in \mathrm{Perf}(X)$ - compact generator

1) smoothness X - smooth $/k$ $\Delta \subset X \times X$ $\mathcal{O}_\Delta \in \mathrm{Perf}(X \times X)$

Def: A/k is smooth if $A \in \mathrm{Perf}(A \otimes A^{op})$. - works for all small dg cat.

Prop: X - smooth $\Leftrightarrow$ A_X is smooth $D(X \times X) \simeq D(A_X \otimes A_X^{op})$

$E \boxtimes E^\vee$

If a small dg category is smooth it is equivalent to a dg algebra.

2) X - proper $\Leftrightarrow$ $\forall E, F \in \mathrm{Perf}(X)$ $\dim \mathrm{Hom}(E, F) < \infty$ ($\Leftarrow$ difficult)

Def: A is proper if $\dim H^\cdot(A) < \infty$. (compact)

Prop: X is proper $\Leftrightarrow$ A_X is proper.

Prop: 1) X - smooth $\Rightarrow$ $\forall\, W: X \to k$ $MF_{coh}(X, W)$ - smooth $/ k[u^{\pm 1}]$ $\deg u = 2$ even $\mathbb{Z}/2$ graded category

2) $\mathrm{Crit}(W) \cap W^{-1}(0)$ proper $\Rightarrow$ $MF_{lf}(X, W)$ proper. ($\Leftrightarrow$ if X - smooth)

either singular on X or critical for W. locally free

$\mathrm{Perf}(A \otimes A^{op}) - MF_{\underset{X^0\times X^0}{L.f}}(X\times X, W \boxminus W)$

$\quad \Psi$

$A \longleftrightarrow \Gamma_{X^0\times X^0}\, \mathcal{O}_\Delta$

diagonal bimodule

Thm (Rouquier) $D^b_{coh}(X)$ has a strong generator.

$E \in \mathcal{T}$ is a strong generator if $\quad \mathcal{T}_0 = \langle E \rangle^{add}$ - direct summands of finite direct sums of shifts of E

$\mathcal{T}_{n+1} = \langle \mathcal{T}_n * \mathcal{T}_0 \rangle = \langle \{ F \mid F_n \to F \to F_0 \to F_n[1] \text{ distinguished}, F_n \in \mathcal{T}_n, F_0 \in \mathcal{T}_0 \} \rangle^{add}$

Karoubi completion.

$\underline{\mathcal{T}_n = \mathcal{T}}$ for some $n \gg 0$.

Prop: A - smooth$/k \Rightarrow A$ is a strong generator of $\mathrm{Perf}(A)$ - follows from the resolution of diagonal bimodule.

Thm: $\forall\, W: X \to k \quad MF_{coh}(X, W)$ has a strong generator.

Thm (Lunts) k-perfect $\Rightarrow D^b_{coh}(X)$ smooth $/k$.

Thm'. X_{red} has a smooth finite stratification $/k \Rightarrow D^b_{coh}(X)$ smooth.

(if k is perfect then regular $\Rightarrow$ smooth; $\mathrm{Spec}(k\sqrt[p]{a})$ is regular not smooth)

Thm: $X/k \qquad W \longrightarrow \mathbb{A}^1_t \qquad$ (i) X^0 has smooth stratification $/k$

(ii) $X \times_{k[t]} k(t)$ has a smooth stratification $/k(t)$.

Then 1) $MF_{coh}(X, W)$ - smooth $/k[u^{\pm 1}]$, 2) $\{ t_0 \mid MF_{coh}(X, W - t_0) \neq 0 \}$ is finite.

(Sard lemma, Bertini theorem).

Idea of proof: $D^b_{coh}(X\times X) \subset \mathrm{Perf}(A \otimes A^{op}) \qquad G \in D^b_{coh}(X)$ - dualizing complex

$R\mathrm{Hom}(-, G): D^b(X) \to D^b(X)^{op} \qquad$ smooth stratification $\Rightarrow \mathcal{E} \boxtimes R\mathrm{Hom}(\mathcal{E}, G)$ is a generator

First we prove 2) and then $\quad A \longleftrightarrow \Delta_* G \Rightarrow 1)$

Strong generator of $\mathrm{Perf}(X) \Leftrightarrow X$ is regular.

Ex. $k(t)$ smooth$/k$. but not finitely generated.

($A \in \mathrm{Perf}(A \otimes A^{op})$ is preserved under)

$f: A \to B$ DG localization $\quad$ (i) $f^*: D(A) \to D(B)$ localization

(ii) $f_*: D(B) \to D(A)$ fully faithful

(iii) $B \overset{L}{\underset{A}{\otimes}} B \to B$ quasi-iso

(iv) $(f \otimes f^{op})^* A \to B$ quasi-iso.

Prop: $f: A \to B$ DG localization, A-smooth $\Rightarrow$ B-smooth.

$k[t] \to k(t)$ - DG localization.

Def: A DG algebra A over k is <u>homotopically finitely presented</u> if A is a homotopy retract of a finite cellular DG algebra $B \overset{gr}{=} k\langle x_1, \dots x_n \rangle$

$dx_i \in k\langle x_1, \dots x_{i-1} \rangle$

$Ho(dg\text{-}alg_k)$ - localization w.r.t. quasi-iso

A is a homotopy retract of B if $\exists$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "id"'] & B \arrow[r, "g"] & A \end{tikzcd}$$
in $Ho(dg\text{-}alg_k)$

if it is additive then B is a direct summand.

SSet	Mod-A	DG-alg$_k$	initial $* \to X$
finite ssets	twisted complexes = semifree finitely generated	finite cellular	is a composition of a finite number of generating cofibrations
finitely dominated = {homotopy retracts of finite ssets}	$Perf(A)$ = {direct summands of twisted complexes}	homotopically finitely presented dg-algebras	

homotopy compact objects

Thm (Töen) 1) h f p $\Rightarrow$ smoothness (finite cellular algebras are smooth).

2) smooth, proper $\Rightarrow$ h f p

3) $D(A) \simeq D(B) \Rightarrow (A \text{ hfp} \Leftrightarrow B \text{ hfp})$

4) A - hfp $E \in Perf(A) \Rightarrow Perf(A)/_{\langle E \rangle}$ - h f p.

Def: A dg algebra (category) A <u>has a smooth compactification</u> if $Perf(A) \simeq (\mathcal{C}/\langle F \rangle)^c_k$ $\mathcal{C}$ - smooth proper

Karoubization.

Thm: char $k = 0$ 1) $D^b_{coh}(X)$ has a smooth compactification

2) $\forall\, W: X \to k$ $MF_{coh}(X, W)$ has a smooth compactification$/ k[u^{\pm 1}]$.

Thm: 1) A/k - associative algebra, k-commutative ring. A is hfp $\Leftrightarrow$ A is smooth (as a DG algebra) and of finite type (finitely presented) $A = {}^{k\langle x_1 .. x_n \rangle}/(f_1 .. f_m)$

2) A/k - commutative dg-algebra $\Rightarrow$ (A hfp $\Leftrightarrow$ A finitely presented, $\mathbb{L}_{A/k} \in Perf(A)$)

cotangent complex over k

Efimov II:

X - separated, of finite type $/k$ $W: X \to \mathbb{A}^1$

$MF_{coh}(X, W)$ obj: $\mathcal{F}_0, \mathcal{F}_1 \in Coh(X)$ $\delta^0: \mathcal{F}_0 \to \mathcal{F}_1$ $\delta^1: \mathcal{F}_1 \to \mathcal{F}_0$

$$\delta^1 \delta^0 = W \cdot id_{\mathcal{F}_0} \quad \delta^0 \delta^1 = W \cdot id_{\mathcal{F}_1}.$$

$MF^{naive}_{coh}(X, W)$ $Hom((\mathcal{F}, \delta), (\mathcal{F}', \delta')) = (Hom(\mathcal{F}, \mathcal{F}'), \delta'(-) \mp (-)\delta)$ -

- $\mathbb{Z}/2$ graded DG-category

$D^b(Coh X) = Ho^b(Coh X)/Acyclic$

$MF_{coh}(X, W) = MF^{naive}_{coh}(X, W)/coAcyclic$

$coAc = \{ Tot(0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0) \mid 0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ distinguished in $MF^{naive}_{coh}(X, W) \}$

zero fiber of X

Th (Orlov): $MF(X, W) \simeq D_{sg}(X^0) = D^b_{coh}(X^0)/D^{perf}(X^0)$ if X-smooth, $W \neq 0$.

$\exists$ finite number $t_0 \in \mathbb{A}^1$ s.th. $MF(X, W - t_0) \neq 0$ (critical values of W).

Th (Positselski) $MF_{coh}(X, W) \simeq D^b_{coh}(X^0)/\langle j^* D^b_{coh}(X) \rangle$ $j: X^0 \hookrightarrow X$, W is not a zero divisor.

Thm: char $k = 0 \Rightarrow D^b_{coh}(X)$ has a smooth compactification, i.e. $F: \mathcal{C} \to D^b_{coh}(X)$

$\mathcal{C}$ - smooth, proper, F-localization in the sense of Drinfeld

Thm (Neeman): (i) $Perf(\mathcal{A}) \xrightarrow{f^*} Perf(\mathcal{B})$ - localization up to direct summands ($f: \mathcal{A} \to \mathcal{B}$)

(localization onto image, everything is a direct summand of image).

(ii) $D(\mathcal{A}) \xrightarrow{f^*} D(\mathcal{B})$ - localization, $Ker f^* = \langle Ker f^* \cap Perf \mathcal{A} \rangle$.

(i) $\Leftrightarrow$ (ii).

If $\mathcal{A}, \mathcal{B}$ - dg algebras $\Rightarrow$ (i) $\mathcal{B} \overset{L}{\underset{\mathcal{A}}{\otimes}} \mathcal{B} \xrightarrow{\sim} \mathcal{B}$

$X = \mathrm{Spec}\, A \qquad D = \{f = 0\} \qquad U = X \setminus D. \qquad j : U \hookrightarrow X \quad j^* : D(X) \to D(U)$

j^* is a localization because $j^* j_* = id$

$\ker j^* = \langle \mathrm{Cone}(A \xrightarrow{f} A) \rangle$ if $G \in \ker j^*$ is right orthogonal to $\mathrm{Cone}(A \xrightarrow{f} A)$

$$f : G \xrightarrow{\sim} G, \ \mathrm{Supp}(G) \subset D \Rightarrow G = 0$$

$\Rightarrow$ $j^* : \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is a localization up to direct summands.

$Z = \{f_1 = \ldots = f_m = 0\} \qquad U = X \setminus Z \qquad j : U \to X$

$\ker j^* = \langle \bigotimes_{i=1}^{m} \mathrm{Cone}(A \xrightarrow{f_i} A) \rangle$ Koszul complex

it works for X - quasi compact, quasi separated, Z given by finite number of equations.
(arguments like in Bondal, Van den Bergh paper).

Y with rational singularities $\quad X \xrightarrow{f} Y$ resolution $, f_* \mathcal{O}_X = \mathcal{O}_Y \Leftrightarrow$

$Lf^* : D(Y) \to D(X)$ is fully faithful $\Leftrightarrow Rf_* : D(X) \to D(Y)$ - localization

unbounded derived categories of quasi-coherent sheaves.

$Rf_* : D^{\pm}_{coh}(X) \to D^{\pm}_{coh}(Y)$ - localization $\quad (f^*$ for $-$, $f^!$ for $+)$ are fully faithful, adjoint

<u>Conj (Bondal, Orlov)</u> (*) $Rf_* : D^b_{coh}(X) \to D^b_{coh}(Y)$ - localization. No adjoint!
2) $\mathrm{Ker}(Rf_*)$ is generated by one object. $\quad \underline{Lf^* D^b_{coh}(Y) \not\subset D^b_{coh}(X)}$

<u>Thm</u>. Y - cone over a smooth Fano hypersurface, $\overset{i:D}{X} = Bl_0 Y$. Then

$Rf_* : D^b_{coh}(X) \to D^b_{coh}(Y)$ is a localization and $\mathrm{Ker}(Rf_*) = \langle i_* (\mathcal{O}_D)^{\perp} \rangle$.

<u>Rmk</u>: Conjecture does not depend on the resolution (whether it is true or not).

<u>Rmk</u>: If Y is proper $\to$ (*) a smooth compactification of $D^b_{coh}(Y)$ (X-smooth, proper)

<u>Def (Lunts)</u> <u>Categorical resolution of a scheme Y</u>:

1) smooth DG category $\mathcal{C}$ and
2) $\pi^* \mathrm{Perf}(Y) \hookrightarrow \mathrm{Perf}(\mathcal{C})$ fully faithful DG functor.
3) $R\pi_* : D(\mathcal{C}) \to D(Y)$ right adjoint $\Rightarrow \mathrm{Im}(\mathrm{Perf}(\mathcal{C})) \subset D^b_{coh}(Y)$

$\Big\}$ ($\mathcal{C} = D^b_{coh}(Y)$ is ok).

4) $\mathrm{Perf}(\mathcal{C}) = \langle D^b_{coh}(X_1), \ldots, D^b_{coh}(X_n) \rangle \quad X_i$ - smooth, $D^b_{coh}(X_i) \subset \mathrm{Perf}(\mathcal{C})$ - left admissible.

<u>Thm</u> (Кузнецов, Лунтс) $\forall\, Y$ sep. f.t./k char $k = 0$ a categorical resolution exists. Moreover, if Y is proper then so are X_i. In particular $\mathcal{C}$ is smooth and proper.

X_i - either a resolution of Y_{red} or a smooth center of a blow-up in the Hironaka process.

Hironaka:

$$Y_m \to Y_{m-1} \to \cdots \to Y_2 \to Y_1 \to Y_0 = Y$$

$Y_{i+1} = Bl_{Z_i} Y_i$ $\quad Z_i \subset Y_i$ smooth, $(Y_m)_{red}$ is smooth $\quad (Bl_Z Y)_{red} = Bl_Z(Y_{red})$

Assume: Y_{red} - smooth, $Y = \operatorname{Spec} A$ $\quad I \subset A$ - nilpotent radical. $(A, I, r : I^r = 0)$.

$Ob(\mathcal{C}) = \{X_1, \dots, X_r\}$ $\quad Hom_{\mathcal{C}}(X_i, X_j) = I^{\max(j-i,0)}/I^j$ - DG category with zero differential

$$End(\mathcal{C}) = \bigoplus_{i,j} Hom_{\mathcal{C}}(X_i, X_j)$$

$Ob(A) = \{\cdot\}$ $\quad End(\cdot) = A$ $\quad A \to \mathcal{C}$ $\quad \cdot \mapsto X_r$ fully faithful

$\pi^* : Perf(A) \to Perf(\mathcal{C})$ - it is a categorical resolution.

$$D^b_{f.g.}(\mathcal{C}) = \langle \underbrace{D^b_{f.g}(A/I), \dots, D^b_{f.g}(A/I)}_{r} \rangle \qquad mod_{f.g}\text{-}A = mod_{f.g}\, \mathcal{C}/\ker \pi_*.$$

with finitely generated cohomology

A/I - smooth $\Rightarrow D^b_{f.g.}(A/I) = Perf(A/I)$.

Prop: π^* gives a categorical resolution when A/I is smooth. π_* is a localization.

$\tilde{Y}$ - ringed space

Prop: $\pi : \tilde{Y} \to Y$ is a categorical resolution, $\pi_* : D^b(\tilde{Y}) \to D^b(Y)$ - localization.

Thm (Efimov): For any Y separated of finite type$/k$ $\exists$ a categorical resolution

$\pi : \mathcal{C} \to Y$ s.th. $\pi_* : Perf(\mathcal{C}) \to D^b_{coh}(Y)$ is a localization.

$Y \subset \bar{Y}$ compactification $\quad D^b_{coh}(Y) = D^b_{coh}(\bar{Y}) / D^b_{coh}(\bar{Y} \setminus Y)$.

Efimov III: 12.10.2012

ringed space, not locally ringed (more than 1 max ideal at each point)

$(S, I, n) \rightsquigarrow \tilde{S} \xrightarrow{\rho_S} S$, $I^n = 0$

1) $\rho_S^*: \mathrm{Perf}(S) \to \mathrm{Perf}(\tilde{S})$ fully faithful

2) $\rho_{S*}: D^b_{coh}(\tilde{S}) \to D^b_{coh}(S)$ localization, $\ker \rho_{S*}$ is generated by 1 object

3) $D^b_{coh}(\tilde{S}) = \langle \underbrace{D^b_{coh}(S_0), \dots, D^b_{coh}(S_0)}_{n} \rangle$

because it comes from localization on abelian categories

we had a quiver $\bullet \to \bullet \to \dots \to \bullet\, \mathcal{O}_S$, the localization $\downarrow$ $\bullet\, \mathcal{O}_S$

$S_0 = (S^{top}, \mathcal{O}_S/I)$ - separated scheme of finite type.

$\left((S_1, I_1, n) \quad (S_2, I_2, n) \quad f: S_1 \to S_2 \quad f^{-1}(I_2) \subset I_1 \right) \to \tilde{S}_1 \xrightarrow{\tilde{f}} \tilde{S}_2.$

Prop: $f: S_1 \to S_2$ proper $\Rightarrow$ $D^b_{coh}(\tilde{S}_1) \xrightarrow{\tilde{f}_*} D^b_{coh}(\tilde{S}_2)$ compatible with decomposition

$$D^b_{coh}(\tilde{S}_1) = \langle D^b((S_1)_0), \dots, D^b((S_1)_0) \rangle \xrightarrow{\langle (f_0)_*, \dots, (f_0)_* \rangle} \langle D^b((S_2)_0), \dots, D^b((S_2)_0) \rangle = D^b_{coh}(\tilde{S}_2)$$

$$\begin{tikzcd} X = Bl_Z Y \arrow[d, "f"] & T = f^{-1}(Z) \arrow[l, hook', "j"'] \arrow[d, "p"] \\ Y & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

smooth

$f_*: D^b(X) \to D^b(Y)$ is not a localization.

It can be corrected to a localization.

$f_*(I_T^n) = I_Z^n$ for $n \gg 0$. (by Serre theorem - it is just a line bundle, for big n there is no higher cohomology).

$\forall n \geq n_0 > 0$.

$$\begin{tikzcd} X = Bl_Z Y \arrow[d, "f"] & T = f^{-1}(S) \arrow[l, hook', "j"'] \arrow[d, "p"] & \tilde{T} \arrow[l, "\rho_T"'] \arrow[d, "\tilde{p}"] \arrow[lld, "\tilde{q}"'] \\ Y & S = Z_{n_0} \arrow[l, hook', "i"'] & \tilde{S} \arrow[l, "\rho_S"'] \end{tikzcd}$$

$\tilde{S}$ ← smooth

$S = Z_{n_0}$ ← neighbourhood

$$\begin{pmatrix} \mathcal{A} & M \\ 0 & \mathcal{B} \end{pmatrix}.$$

$D = \langle D^b(\tilde{S}), \mathrm{Perf}(X) \rangle$, gluing $\tilde{p}_*(j\rho_T)^*$ - glueing given by a ~~bimodule~~ functor = bimodule

$\pi^*: \mathrm{Perf}(Y) \to D$

$\pi^*(E) = ((i\rho_S)^* E, f^* E, \mathrm{adj})$ ← glueing is the adjunction map.

Prop: π^* is fully faithful - using projection formula we get right adjoint.

$\mathrm{Tor\,dim}(j\rho_T) = 1 \Rightarrow \tilde{p}_*(j\rho_T)^* D^b_{coh}(X) \subset D^b_{coh}(\tilde{S})$.

$D \subset D_{coh} = \langle D^b_{coh}(\tilde{S}), D^b_{coh}(X) \rangle$, gluing $\tilde{p}_*(j\rho_T)_*$

$\mathcal{C} \xrightarrow{\pi_X} X$ resolution of sing

$\mathcal{C}' = \langle D^b(\tilde{S}), \mathcal{C} \rangle \to Y$ resolution of sing., gluing $\tilde{p}_*(j\rho_T)^* \pi_{X*}$

Thm: $\pi_*: D_{coh} \to D^b(Y)$ is a localization, $\ker \pi_*$ is generated by 1 object

$$D_{coh} \ni (F_X, F_{\tilde{S}}, \mu_F: \tilde{p}^* F_{\tilde{S}} \to (j\rho_T)^* F_X) = F$$

Fiber = cone[-1] $\quad \pi_*(F) = \mathrm{Fiber}\left(f_* F_X \oplus (i\rho_S)_* F_{\tilde{S}} \to \tilde{q}_*(j\rho_T)^* F_X \right)$

$\pi_* |_{D^b(\tilde{S})} = (i\rho_S)_* \qquad \ker \rho_{S*} \subset \ker \pi_*$

$$\begin{tikzcd} D_{coh}/\ker \rho_{S*} = \langle D^b(S), D^b(X) \rangle \arrow[r, phantom, "="] & D^!_{coh} \arrow[l, bend left, "\rho_j^{*,*}"] \end{tikzcd}$$

$$D^!_{coh} \xrightarrow{\pi^!_*} D^b(Y).$$

T is a Cartier divisor

$\phi : D^b(T) \longrightarrow \ker \pi^!_*.$ $\quad \phi(G) = (j_* G, \; p_*(G[T])[1], \; \mu_G)$

$$(-T)[1] \to j^* j_* \to id \to (-T)[2]$$

ϕ(Generator) = Generator.

<u>Thm:</u>

$$\begin{tikzcd} Bl_S Y = X \arrow[d, "f"] & T \arrow[l, hook', "j"'] \arrow[r, hook, "j_{12}"] \arrow[d, "p = p_1"] & T_2 \arrow[r, hook] \arrow[d, "p_2"] & \cdots \arrow[r, hook] & T_n \arrow[d, "p_n"] \\ Y & S \arrow[l, hook', "i"'] \arrow[r, hook, "i_{12}"] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \end{tikzcd}$$

Assume that $f_* \mathcal{O}_X = \mathcal{O}_Y$, $p_{n*} \mathcal{O}_{T_n} = \mathcal{O}_{S_n}$ $\forall n \geq 1$.

Then if $p_* D^b(T) \to D^b(S)$ is a localization $\Rightarrow$ so are all p_{n*} and f_*.

$\ker f_* = \langle j_* \ker p_* \rangle$, $\ker p_{n*} = \langle i_{1,n} \ker p_* \rangle$

Neeman:

$\mathrm{Perf}(\mathcal{A}) \xrightarrow{g^*} \mathrm{Perf}(\mathcal{B})$ is a localization $\Leftrightarrow$ (i) $D(\mathcal{A}) \xrightarrow{Lg^*} D(\mathcal{B})$ is localization,

(ii) $\ker Lg^* = \langle \ker Lg^* \cap \mathrm{Perf}\,\mathcal{A} \rangle$

(iii) $\mathrm{Im}\, g^*$ generates $\mathrm{Perf}(\mathcal{B})$ without Karoubi completion.

needed in one place

$D(D^b_{coh}(D)) = D^{co}(X)$ - coderived category

D^b_{coh} ↑ DG-category $\quad D^{co}(X) = K(\mathrm{inj}_X)$. Krause, Positselski:

$D^{co}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} D^{co}(Y)$ is a localization $\quad f_* f^! = id.$

$D^b_T(X) \longrightarrow D^b_S(Y)$ - localization $\Rightarrow D^{co}(X) \to D^{co}(Y)$ is a localization,

$\ker(D^{co}(X) \to D^{co}(Y)) \subset D^{co}_T(X)$

$= \ker(D^{co}_T(X) \to D^{co}_S(Y))$, where $D^{co}_T(X) = D(D^b_T(X))$.

$D^b(coh_T X) = D^b_T(coh\, X) = \mathrm{hocolim}\, D^b(T_n)$ because $coh_T X = \mathrm{colim}_n\, coh\, T_n$

<u>Lemma:</u>

$$\begin{tikzcd} \mathcal{A}_i \arrow[r] & \mathcal{A}_{i+1} \\ \mathcal{B}_i \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{B}_{i+1} \arrow[u] \end{tikzcd}$$

DG-cat.

$\Rightarrow \mathrm{hocolim}(\mathcal{A}_i/\mathcal{B}_i) \xrightarrow[\text{quasi-equiv.}]{\sim} \mathrm{hocolim}(\mathcal{A}_i)/\mathrm{hocolim}(\mathcal{B}_i)$

$\Rightarrow p_{n*}$ - localization $\Rightarrow f_* : D^b_T(X) \to D^b_S(X)$ - localization, $\ker f_* = \langle j_{n*}(\ker p_{n*}), n \geq 1 \rangle$

<u>Lemma:</u> Suppose that $p_{n*} : D^b(T_n) \to D^b(S_n)$ is a localization. Then $p_{n+1} : D^b(T_{n+1}) \to D^b(S_{n+1})$ is a localization. $\ker p_{n+1} = \langle j_{n,n+1\,*} \ker p_{n*} \rangle$

To prove it we need Auslander construction.

$(T_{n+1}, I^n_T / I^{n+1}_T, 2)$

$(S_{n+1}, I^n_S / I^{n+1}_S, 2)$

$$\begin{tikzcd} T_n \arrow[r, hook, "j_{n,n+1}"] \arrow[d, "p_n"] & T_{n+1} \arrow[d, "p_{n+1}"] & \tilde{T}_{n+1} \arrow[l, "\rho_{T_{n+1}}"'] \\ S_n \arrow[r, hook, "i_{n,n+1}"] & S_{n+1} & \tilde{S}_{n+1} \arrow[l, "\rho_{S_{n+1}}"'] \end{tikzcd}$$

$\rho_{T_{n+1}*}$ - localization, $\rho_{S_{n+1},*}$ - localization, p_{n*} - localization.

p_{n+1}, $\tilde{p}_{n+1 *}$ - localizations $\Rightarrow$ $p_{n+1 *}$ is a localization.

$$\begin{array}{ccc} D^b(\widetilde{T_{n+1}}) = & \langle D^b(T_n), & D^b(T_n)\rangle \\ {\scriptstyle \tilde{p}_{n+1 *}}\downarrow & {\scriptstyle p_{n*}}\downarrow & \downarrow{\scriptstyle p_{n*}} \\ D^b(\tilde{S}_{n+1}) = & \langle D^b(S_n), & D^b(S_n)\rangle \end{array}$$

by Neeman $p_{n+1 *}$ - localization.

$\ker p_{n+1 *} = \rho_{T_{n+1}}^*(\ker(\rho_{S_{n+1}*}\tilde{p}_{n+1 *})) = \rho_{T_{n+1}}^*(\tilde{p}_{n+1 *}^{-1}(\ker p_{S_{n+1}}))$

$\exists$ natural functor $D^b(S_n) \xrightarrow{\Psi} \ker \rho_{S_{n+1}}$, $D^b(T_n) \xrightarrow{\Psi_1, \Psi_2} D^b(\widetilde{T_{n+1}})$.

$\operatorname{Im}\Psi_1, \operatorname{Im}\Psi_2$ generate $\ker$ $\quad \rho_{T_{n+1}}\Psi_i = j_{n,n+1}$.

For general statement $f_* \mathcal{O}_X \to \mathcal{O}_Y$ we need glueing.